

THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2230 Tutorial 4
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Definition 1. Let $w(t) = u(t) + iv(t)$ be a complex function of a real variable t , the definite integral of $w(t)$ over the interval $a \leq t \leq b$ is defined as

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt$$

Definition 2. Let $z(t) = x(t) + iy(t) : [a, b] \rightarrow \mathbb{C}$ be a continuous complex function of a real variable t , $z(t)$ is a **simple curve** or **Jordan curve** (path or curve) if $z(t)$ is one to one (the curve does not intersect itself). It is **closed** if $z(a) = z(b)$. Such a closed curve is positive oriented when it is in the counterclockwise direction.

Definition 3. A **contour** is a piecewise smooth simple curve.

Remark: Sometime we may require a contour to be piecewise differentiable.

Definition 4. Let f be piecewise continuous on a contour C represented by $z(t) : [a, b] \rightarrow \mathbb{C}$. The line integral (contour integral) of f along C is defined to be

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

Definition 5.

$$\int_C f(z)|dz| = \int_a^b f(z(t))|z'(t)|dt$$

Proposition 1.

$$\left| \int_C f(z)dz \right| \leq \int_C |f(z)||dz|$$

Example 1. Evaluate the integral with the principal branch

$$\int_C z^{-1+i}dz$$

where C is the positively oriented unit circle.

To parametrize a circle, we have $z(\theta) = e^{i\theta}$ for $\theta \in (-\pi, \pi]$.

$$f(z)dz = f(z(\theta))ie^{i\theta}d\theta = e^{(-1+i)\theta}ie^{i\theta}d\theta = e^{-\theta}id\theta$$

$$\int_C z^{-1+i}dz = \int_{-\pi}^{\pi} e^{-\theta}id\theta = i(-e^{-\pi} + e^{\pi})$$

we should be careful that z^{-1+i} is not defined on the branch cut $\{arg(z) = \pm\pi\}$ but z^{-1+i} is still piecewise continuous on C .

Definition 6. Suppose C is a contour represented by $z(t) : [a, b] \rightarrow \mathbb{C}$, then the length of the contour is the integral

$$L = \int_a^b |z'(t)| dt$$

In \mathbb{R}^2 , the line integral may be independent of the path taken (only depend on the two ends of the path), we would wonder if it is true for contour integral in \mathbb{C} .

Theorem 1. Suppose that $f(z)$ is continuous in a open connected set D . The following statements are equivalent

- $f(z)$ has an antiderivative $F(z)$ throughout D ($F'(z) = f(z)$)
- Given any two fixed points z_1 and z_2 in D , for any contour lying in D with end points z_1 and z_2 , the contour integral has a fixed value depends only on z_1 and z_2 (path independent)
- the contour integrals of $f(z)$ along any closed contours lying entirely in D all have value zero.

Moreover, if $f(z)$ has an antiderivative $F(z)$, then

$$\int_C f(z) dz = \int_{z_1}^{z_2} f(z) dz = F(z_2) - F(z_1)$$

Remark : We should consider $f(z) = 1/z$. It seems that the antiderivative of f is $F(z) = \log z$, however $\log z$ is not well-defined at the branch cut (in the principal branch, $\log z$ is not well-defined at the ray $\arg(z) = \pm\pi$ and it can not be differentiable there). Hence $f(z) = 1/z$ does not have antiderivative in D . You can compute directly that $\int_{|z|=1} \frac{dz}{z} = 2\pi i$ which is not zero.

Exercise:

1 Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant, show that $\left| \int_C \frac{z+5}{z^2-1} dz \right| \leq 7\pi/3$.

3 Let C be the of the circle $|z - 1| = 2$, compute $\int_C \frac{z dz}{z - 1}$.

4. Compute the integral $\int_C f(z) dz$ with

(a) C is the arc of the semicircle $z = 2e^{i\theta}$ ($0 \leq \theta \leq \pi$) and $f(z) = \frac{z+2}{z}$